Passive Circuits and Devices

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Contents

1 Introduction

These notes are meant as an introduction to circuit design using the standard passive components: resistors, capacitors, and inductors. The reader is expected to have background in calculus and some electromagnetism is also helpful. The topics covered here make up the first half of Harvard's ES152: Circuits, Devices, and Transduction. The second half of the class covers active circuits and devices, including op-amps, basic semiconductor physics, PN-junctions, BJTs, MOSFETs, and single transistor amplifiers.

A circuit is a collection of devices that are interconnected to perform some function. Examples include a light switch, a computer, an audio speaker, and much more. We will generally split circuits into two classes: analog and digital.

Analog circuits deal with continuous signals, usually sensed from the real world. We may wish to amplify a signal such as an audio wave sensed from a microphone, or filter a signal to remove noise. Analog circuits can also be used to perform mathematical operations on a waveform or between two waveforms, such as integration or subtraction. For the most part analog circuits are hand-made as attempts to automate design with CAD (computer aided design) tools have largely failed. This is because analog circuits can be hard to specify and can be susceptible to noise (they are not as robust as their digital counterpart). For the most part we will focus on analog circuits.

Digital circuits assign voltage ranges to discrete values such as "0" and "1". This allows for more robustness because slight variations in voltage due to noise won't change the behavior of the circuit. From this binary abstraction, Boolean logic functions can be created in hardware that perform functions like NAND, OR, XOR, ... From these functions we can create logic that can implement any boolean function, including binary multiplication or addition. Digital circuits can also implement memory using components like capacitors or latches (made from feedback with logic gates). From this memory we can create circuits that store internal state and operate based on the current inputs and all previous inputs (they remember a state that was arrived at by the sequence of all previous inputs). Usually digital circuits like this are clocked to simplify the design and ensure that all memory elements are advancing (deciding when to store new data) at the same rate. The robustness of digital circuits is what makes integrated circuits with billions of transistors on board possible.

Mixed-signal circuits try to leverage both types by interfacing digital circuitry with analog circuitry. For example mixed-signal circuits might be used in embedded systems where the overall system is controlled by a digital processor but a certain task is performed by analog circuitry for the benefit of low power and speed. In such cases designers must be willing to accept some noise.

Electrical engineering is an exciting field because of the ability to create extremely complex systems that perform very useful functions. Circuit designers deal with this complexity by using "black boxes." A designer does not need to know the details of how some component works, only what its inputs and outputs are. This is an essential concept in engineering in general.

Integrated circuits?

2 Circuit Terminology

2.1 Charge

Electric charge is a fundamental property of matter. Electric charges can be positive or negative and the minimum charge existing in nature is that of a single electron or proton, denoted by e. The unit of charge is the Coulomb (C), and $e = 1.6 \times 10^{-19}$C. An electron carries a charge, of $-e$ and a proton carries a charge of e. The *net* charge of a closed region is the sum of all the charges in the region, and the region is said to be neutral if the net charge is 0. The law of conservation of charge states that the net charge in a closed region and neither be created nor destroyed. Since all matter is made from protons and electrons the charge of anything must be an integral multiple of e. Charge is usually denoted by the variable q.

2.2 Current

Current is the motion of charge. In a conductor, such as copper, the electrons are free to move around. When they all move in the same general direction a current arises. Specifically the current is the amount of charge dq that crosses a cross-sectional area of the wire in some infinitely small amount of time dt

$$i = \frac{dq}{dt}$$

and the unit for current is C/s also called the Ampere (A). Electrons are not the only kind of charge carrier that can exist. Other charge carriers include ions, for example in water, or "holes" which represent the lack of an electron. Imagine a box of golf balls – if you remove one then another will move into the hole fill its place, creating another hole. This hole travels through the medium and can be thought of as a physical quantity itself. Holes are very important in the study of semiconductors. By convention the current is positive in the direction opposite the flow of electrons, so the current flows from higher potential to lower potential (while electrons are negative and move the other way).

It is often a misconception that the electrons in a circuit move at the speed of light. In fact, they move quite slowly at about 10^{-4}m/s. However as each electron moves a small amount, it pushes its neighbor by a small amount. The propagation of this aggregate motion is very fast and is why we observe electrical signals to propagate near the speed of light.

Example:[1] The charge flowing past a certain location in a wire is given by

$$q(t) = \begin{cases} 0 & t < 0 \\ 5te^{-0.1t} & t \geq 0. \end{cases}$$

1. Determine the current at $t = 0$

We apply the definition of current:

$$\begin{aligned} i = \frac{dq}{dt} = \frac{d}{dt}(5te^{-0.1t}) &= 5e^{-0.1t} - 0.5te^{-0.1t} \\ &= (5 - 0.5t)e^{-0.1t} \text{ A.} \end{aligned}$$

Setting $t = 0$ gives $i(0) = 5$ A.

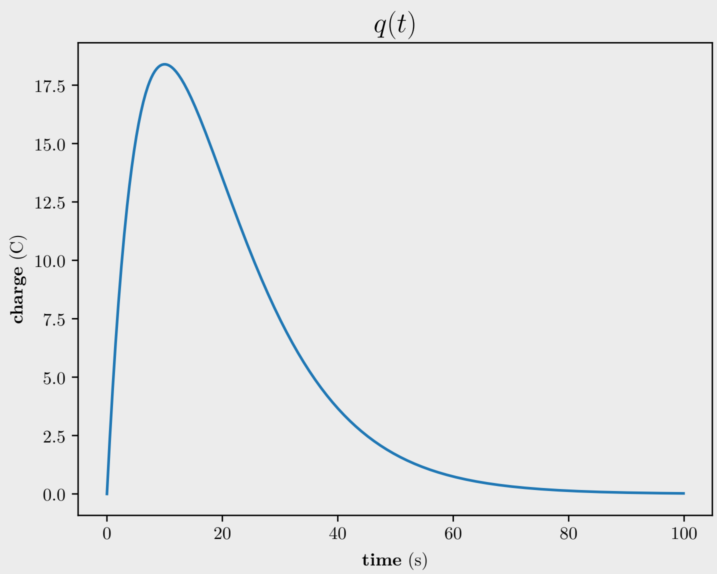
2. What is the instant at which $q(t)$ is maximized and find the corresponding value of q.

To maximize $q(t)$ we take its derivative and set it to 0:

$$\frac{dq}{dt} = (5 - 0.5t)e^{-0.1t} = 0.$$

[1] Example 1-3 in Ulaby

This is satisfied when $t = 10$ or $t = \infty$. By plugging in these values we see $q(\infty) = 0$ and $q(10) = 18.4$ therefore $t = 10$ is a maximum and $t = \infty$ is a minimum (this is also seen by plotting). The final answer is $t = 10$ with $q(10) = 18.4$ C.



2.3 Voltage

Voltage (also called electric potential difference) is a relative quantity measured between two points. Consider a system with two points a and b. The voltage difference v_{ab} from a to b is the amount of energy that is gained or lost per unit charge to move a test charge between a and b. Specifically, the voltage v_{ab} is

$$v_{ab} = \frac{dw}{dq}$$

where dw is the energy in Joules to move a charge dq from b to a. The unit of voltage is the Volt (V). It can be helpful to think of voltage like pressure – pushing against the charge carriers. Pushing on the charges does not necessarily make them move (for example if the circuit is disconnected) or can make them move at a certain rate depending on the medium they are trying to move through. We will see the relationship between voltage, current, and resistance soon in the form of Ohm's law.

Since voltage is only defined between two points, when we refer to something having a certain voltage it is always with respect to some reference, which is usually obvious. For example in the case of a 1.5V battery the voltage is the difference in electric potential between the positive and negative terminals. In circuits, we will define a *ground* terminal and every other voltage will be measured with respect to that terminal. The choice of ground is completely arbitrary but often it will help simplify the math to choose ground in certain places (for example at the negative end of a battery).

2.4 Circuit elements

Circuits are composed of elements and wires that connect them. It will help our analysis to define some circuit terminology, and then introduce some basic circuit devices.

- Node: An electrical connection between two or more elements.
- Ordinary node: An electrical connection node that connects to only two elements.

- Extraordinary node: An electrical connection node that connects to three or more elements.
- Branch: Trace between two consecutive nodes with only one element between them.
- Path: Continuous sequence of branches with no node encountered more than once.
- Extraordinary path: Path between two adjacent extraordinary nodes.
- Loop: Closed path with the same start and end node.
- Independent loop: Loop containing one or more branches not contained in any other independent loop.
- Mesh: Loop that encloses no other loops.
- In series: Elements that share the same current. They have only ordinary nodes between them.
- In parallel: Elements that share the same voltage. They share two extraordinary nodes.

2.5 Sign convention

2.6 Power

2.7 Units and conventions

We will use the SI units, shown for various quantities in the table below.

Dimension	Unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric charge	coulomb	C
Temperature	kelvin	K
Current	ampere	A
Voltage	volt	V
Resistance	ohm	Ω
Capacitance	farad	F
Inductance	henry	H
Power	watt	W
Frequency	hertz	Hz

3 Resistive Circuits

3.1 Resistors

Resistors are the first device we will study. They are simply conductors which are less conductive than wires. The resistance of a block of material can be calculated from three properties and quantifies how difficult it is for current to pass through it.

The three properties are the conductivity σ of the material itself, the length ℓ of the block along which the current is traveling, and the cross-sectional area A of the block through which the current is traveling. Once these values are known the resistance is

$$R = \frac{\ell}{\sigma A} = \rho \frac{\ell}{A}.$$

Sometimes conductivity is swapped for the resistivity ρ defined as $\rho = \frac{1}{\sigma}$. The unit of resistance is the Ohm (Ω).

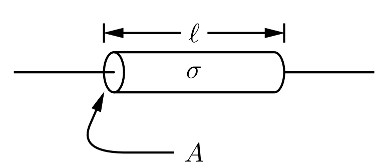


Figure 1: $R = \frac{\ell}{\sigma A}$

Every component in the circuit has a resistance associated with it, even the wires, though we often assume that the wires have zero resistance because their resistance is so much smaller than resistors. In integrated circuits though, interconnect delay plays a significant role and must be taken into account.

Various materials exhibit variable resistance, such as some metal oxides whose resistivity is sensitive to temperature. Such materials can be used to make *thermistors* which are resistors used for measuring temperature. Similar measurements can be made for pressure using *piezoresistors*. Other kinds of variable resistors exist in the form of *rheostats* and *potentiometers* which are resistors whose resistance can be adjusted by turning a knob (usually changing the length of the block and therefore the resistance).

Example: Two concentric cylindrical shells of radius a and b and length ℓ have a material with conductivity σ between them (a hollowed out cylinder).

1. If current flows along the length ℓ of the hollowed cylinder what is the resistance of the medium?

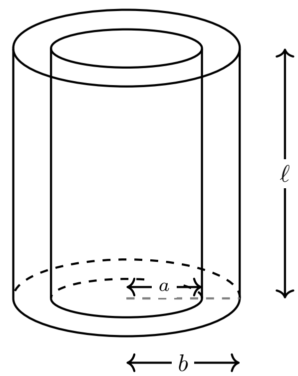


Figure 2: Hollowed cylinder

The cross-sectional area is $\pi(b^2 - a^2)$ and the length is ℓ so the resistance is

$$R = \frac{\ell}{\sigma\pi(b^2 - a^2)}.$$

2. If current flows from the inner cylinder to the outer cylinder (not along the length of the cylinders!), what is the resistance of the medium?

The cross-sectional area the current flows through grows as the current gets farther from the inner cylinder. At some distance r from the center, the cross-sectional area is $2\pi r\ell$. If the current flows through a length dr then the resistance of the shell at r is

$$dR = \frac{dr}{\sigma 2\pi r\ell}.$$

To get the overall resistance of the medium we integrate dR from a to b:

$$R = \int_a^b dR = \int_a^b \frac{dr}{\sigma 2\pi r\ell} = \frac{1}{2\pi\sigma\ell} \ln\left(\frac{b}{a}\right).$$

3.2 Ohm's law

Ohm's law describes the $i - v$ characteric of the resistor. Specifically it states that the voltage v across a resistor is linearly proportional to the current i through the resistor, and the proportionality factor is the resistance R:

$$v = iR.$$

Ideal resistors exhibit this linear behavior, but in practice the $i - v$ curve of a resistor is only linear within a finite range. When we use Ohm's law we implicitly assume that the resistor is operating within this range.

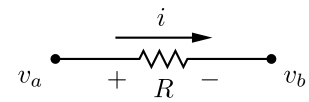


Figure 3: Current enters the "+" side, giving $i = \frac{v_a - v_b}{R}$.

When you think about it, it is a bit strange that the relationship is linear. After all, we expect that the electric field in the conductor will exert a force on the electrons, and therefore provide acceleration, resulting in a quadratic relationship. However, because of thermal and other noise the electrons only have a net velocity in one direction, but individually they are constantly bumping into each other. Every time an electron bumps into another one or a wall, the acceleration is stopped. This provides an intuitive explanation for why the relationship is linear.

3.3 Power loss

Pushing current through a resistor takes energy, which eventually manifests itself as heat. If a charge q is pushed at an average velocity $\mathbf{v}$ by an electric field $\mathbf{E}$ then the rate at which work is done is $q\mathbf{E} \cdot \mathbf{v}$. The reason this manifests as heat is related to the discussion from before about the charge carriers constantly scattering.

Recall that current is charge per time. Then if we have a current i flowing through a resistance R over a time period of Δt the total charge that passes through the resistor is $i\Delta t$. As we said before, voltage is the amount of work per unit charge to move between two points, so if the voltage across the resistor is v then the work done in time Δt is $(i\Delta t)v$. The rate at which the work is done, known as power, is therefore

$$p = iv = i^2 R.$$

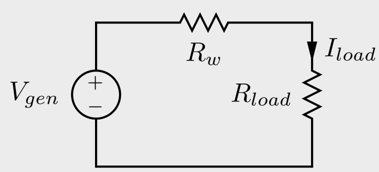
This can also be derived through the use of the chain rule:

$$p = \frac{dw}{dt} = \frac{dw}{dq}\frac{dq}{dt} = vi.$$

The unit of power is the watt (W). Resistors are usually given a power rating, which specifies the amount of power a resistor can withstand before burning up. A typical resistor might have a power rating of 0.25 W.

Example: Power is usually generated remotely and delivered to cities via large cables at a high voltage. Why is a high voltage necessary to deliver the power? Consider a 100km long, 5mm thick copper cable at a voltage of 120V. If a house is dissipating 1.2kW of power from the 120V DC source, what is the power lost in the cable?

We can model the system with the following circuit:



Since we know the power and the voltage, we can find the current drawn by the house to be

$$I_{load} = P/V = 1\text{kW}/120\text{V} = 10\text{A}.$$

Now we find the resistance R_w. We know the length of the wire, the cross-sectional area, and conductivity of copper, thus

$$R_w = \rho\frac{\ell}{A} = \rho\frac{\ell}{\pi r^2} = 1.72 \times 10^{-8} \times \frac{100 \times 10^3}{\pi(5 \times 10^{-3})^2} = 22\Omega.$$

This may not seem like much resistance, but driving 10 amps will lead to a power loss to heat of

$$P_{loss} = I_{load}^2 R_w = 2.2\text{kW}.$$

This is even more than the power delivered to the house! In reality, power lines use a very high voltage to keep the current low and use efficient transformers to transform the voltage between 120V and a much higher voltage. Once of the reasons we get AC current from the wall is that it is easier to build efficient transformers for AC current (though recently DC transformers have been rapidly improving).

3.4 Kirchoff's laws

Kirchoff's current and voltage laws form the fundamental laws that govern circuit analysis. The two can be used on their own to analyze many circuits, though there are techniques that build on Kirchoff's laws which we will see later that make circuit analysis quicker and easier.

3.4.1 Kirchoff's current law (KCL)

Kirchoff's current law follows from the law of conservation of charge. At any node in a circuit, charge is simply passing through a wire, and therefore the sum of the currents that enter the node must be equal to the sum of currents that exit the node. Alternatively we can say that all currents entering and exiting the node must sum to zero

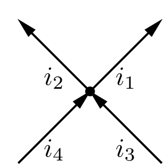


Figure 4: $-i_4 - i_3 + i_2 + i_1 = 0$

$$\sum_{n=1}^{N} i_n = 0$$

where n is the number of branches connected to the node and i_n is the nth current. Currents entering and exiting are assigned opposite signs. Often currents exiting are assigned a "+" and currents entering a "−."

Writing the KCL equations for the nodes in a circuit will give a system of equations that can help to solve the circuit.

Example:[2] If the voltage V_4 across the 4 Ω resistor is 8 V, determine I_1 and I_2.

[2] Example 2-4 in Ulaby

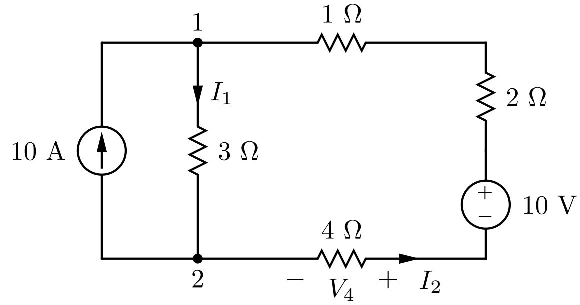


Figure 5: Circuit for KCL example

Since I_2 enters the "−" terminal of the resistor, to be consistent with the passive sign convention we include a negative sign in our Ohm's law relationship:

$$I_2 = -\frac{V_4}{4} = -\frac{8}{4} = -2 \text{ A}.$$

To find I_1 we use KCL at node 2:

$$-I_1 + I_2 + 10 = 0$$

which gives

$$I_1 = 10 + I_2 = 10 - 2 = 8 \text{ A}.$$

3.4.2 Kirchoff's voltage law (KVL)

The law of conservation of energy states that the net energy gained or lost by moving an electric charge in closed loop is zero. Since voltage represents the amount of energy needed to move a charge between two points, around a closed loop the net voltage must be zero:

$$\sum_{n=1}^{N} v_n = 0$$

where N is the number of branches in the loop and v_n is the voltage across the nth branch. An alternative statement is that the total voltage rise around a closed loop must equal the total voltage drop around the loop. When applying KVL the signs can become confusing. The convention we use is the following:

- Assign "+" and "−" sides to unknown voltages in the loop arbitrarily.
- Add up the voltages clockwise around the loop.
- Assign a positive sign to the voltage across an element if the "+" side of the element is encountered first, and a negative sign if the "−" side is encountered first.

Armed with Ohm's law, KCL, and KVL you should now be able to solve many different kinds of circuits. Sometimes it can be difficult to know where to start though. Here is a set of steps you can follow to solve circuits with Kirchoff's laws, and in the next section we will look at more sophisticated techniques built on these concepts that will further help in circuit analysis.

- Identify extraordinary nodes and loops in the circuit, and the N unknowns associated with them.

- Use KCL, KVL, and Ohm's law to write N independent equations
  - Write as many KVL loop equations as you can, adding one additional circuit element per loop. Avoid loops that involve current sources.
  - For the remaining equations you need to solve the system, use KCL, making sure each node picks up an additional current.
- Solve the system of equations using your preferred method.

Example:[3] For the circuit shown below, determine the voltages at V_a and V_b and all branch currents associated with those nodes (I_1, I_2, I_3, I_4).

Assume $V_0 = 10$ V, $I_0 = 0.8$ A, $R_1 = 2\ \Omega$, $R_2 = 3\ \Omega$, $R_3 = 5\ \Omega$, $R_4 = 10\ \Omega$, and $R_5 = 2.5\ \Omega$.

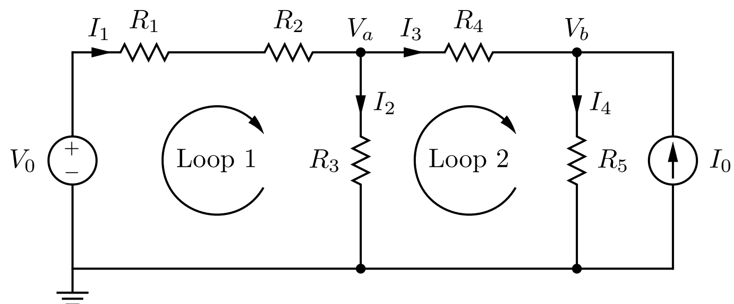


Figure 6: Circuit for KVL/KCL example

We start by writing the KVL loop equations for Loops 1 and 2:

$$-V_0 + I_1R_1 + I_1R_2 + I_2R_3 = 0$$
$$-I_2R_3 + I_3R_4 + I_4R_5 = 0.$$

We cannot write a KVL equation for the loop involving the current source so instead we write two KCL equations for nodes V_a and V_b:

$$-I_1 + I_2 + I_3 = 0$$
$$-I_3 + I_4 - I_0 = 0.$$

We now have four equations and four unknowns so we solve the system and find

$$I_1 = 1.1 \text{ A}, \quad I_2 = 0.9 \text{ A}, \quad I_3 = 0.2 \text{ A}, \quad I_4 = 1 \text{ A}.$$

To find the node voltages we simply take the current and apply Ohm's law

$$V_a = I_2R_3 = 4.5 \text{ V}$$
$$V_b = I_4R_5 = 2.5 \text{ V}.$$

Example:[4] For the circuit given below, determine the amount of power consumed by the 12 Ω resistor.

[3] Example 2-6 in Ulaby
[4] Example 2-8 in Ulaby

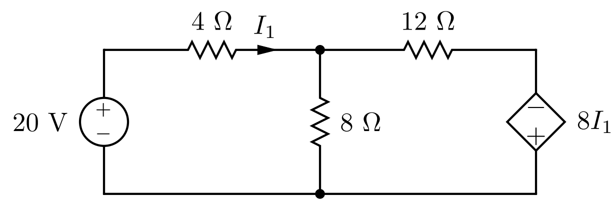
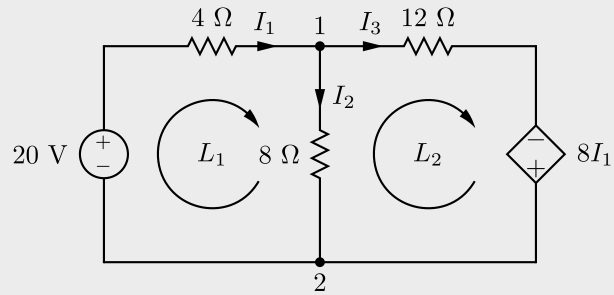


Figure 7: Circuit for KVL/KCL example

We start by assigning currents, nodes, and loops:



For loops 1 and 2 KVL gives us

$$-20 + 4I_1 + 8I_2 = 0$$
$$-8I_2 + 12I_3 - 8I_1 = 0.$$

Now we do KCL at node 1 and get

$$-I_1 + I_2 + I_3 = 0.$$

Solving the system of three equations and three unknowns yields

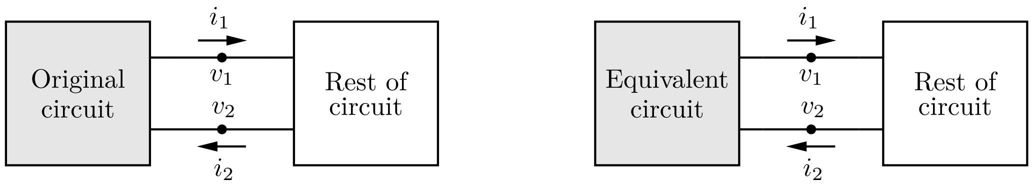
$$I_1 = \frac{25}{7}\text{ A},\quad I_2 = \frac{5}{7}\text{ A},\quad I_3 = \frac{20}{7}\text{ A}.$$

So the power dissipated in the 12 Ω resistor is

$$P = I_3^2 R = \left(\frac{20}{7}\right)^2 \cdot 12 = 97.96\text{ W}.$$

3.5 Equivalent circuits

The concept of equivalent circuits is very important in analysis because it allows us to transform complex circuits into simple ones. Consider a sub-circuit which is connected to the rest of the circuit by some numnber of nodes. Two circuits connected at those nodes are equivalent if the voltage and current at each node is the same for both circuits. In other words, if we replace one sub-circuit with an equivalent circuit then it looks like nothing has changed from the view of the rest of the circuit. We will now examine how to find the equivalent circuit for various configurations. Performing transformations to these equivalent circuits may drastically simplify your analysis.



(a) Original circuit (b) Equivalent circuit

Figure 8: As long as i_1, i_2, v_1 and v_2 are the same for both circuits they are equivalent

We can apply this concept of equivalence to simplify circuits with resistors in series or in parallel. The same can also be done for voltage sources in series and current sources in parallel.

3.5.1 Resistors in series

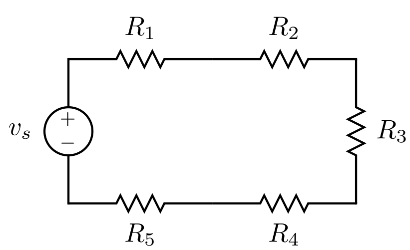
Consider a circuit loop with a voltage source v_s containing n resistors in series. If we write the KVL equation for the loop we see

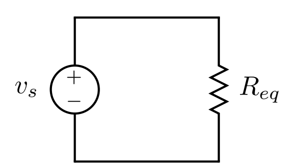
$$-v_s + R_1 i_s + R_2 i_s + \ldots + R_n i_s = 0,$$

which can be written as

$$v_s = \sum_{j=1}^{n} R_j i_s = i_s \sum_{j=1}^{n} R_j = R_{eq} i_s.$$

Hence a circuit with n series resistors is equivalent to a circuit with one resistor with resistance equal to the sum of the series resistors.





(b) Equivalent circuit with $R_{eq} = R_1 + R_2 + R_3 + R_4 + R_5$

(a) Circuit with five series resistors

Figure 9: Resistors in series

A simple extension to this fact is called voltage division. For some resistor R_i we can notice that the voltage drop v_i across it is

$$v_i = R_i i_s = R_i \left(\frac{v_s}{R_{eq}} \right) = \left(\frac{R_i}{R_{eq}} \right) v_s.$$

The circuit below is often called a voltage divider, and the voltage v_2 is given by

$$v_2 = \left(\frac{R_2}{R_1 + R_2} \right) v_s.$$

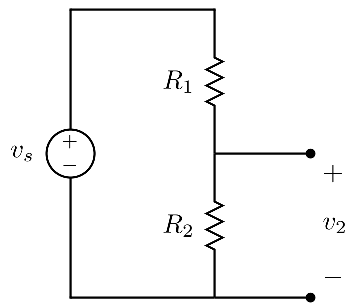


Figure 10: A voltage divider

When a load circuit is connected at the nodes shown it will see a voltage of approximately v_2 as long as the resistance of the load circuit is large compared to R_2.

3.5.2 Resistors in parallel

A similar analysis can be made for resistors in parallel. Notice that the current that goes through each parallel resistor must sum to the total current flowing out of the source. Thus

$$i_s = i_1 + i_2 + i_3$$

and Ohm's law gives

$$i_1 = \frac{v_s}{R_1}, \quad i_2 = \frac{v_s}{R_2}, \quad i_3 = \frac{v_s}{R_3}$$

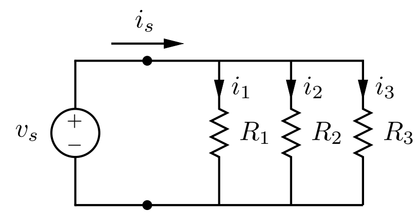


Figure 11: Parallel resistors

which means

$$i_s = v_s \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right).$$

We can replace the three resistors with a single equivalent resistor such that

$$i_s = \frac{v_s}{R_{eq}}.$$

Thus

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

More generally for N parallel resistors, the equivalent resistance is

$$R_{eq} = \left(\sum_{i=1}^{N} \frac{1}{R_i} \right)^{-1}.$$

This is a more unwieldy equation because it's hard to use for quick arithmetic. It is useful to know that for two parallel resistors the equivalent resistance is

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

and for two parallel and equivalent resistors, $R_{eq} = R/2$. We will sometimes denote the equivalent resistance between two parallel resistors with the shorthand $R_1 \| R_2$.

Current division is the property that the current through the jth parallel resistor is

$$i_j = \left(\frac{R_{eq}}{R_j}\right) i_s$$

and for two resistors in parallel we have

$$i_1 = \left(\frac{R_2}{R_1 + R_2}\right) i_s, \quad i_2 = \left(\frac{R_1}{R_1 + R_2}\right) i_s.$$

Example:[5] Consider the circuit below which extends infinitely to the right. This is called an "attenuator chain" or "ladder network." It can be used for dividing the voltage at each step by some amount. For example when $R_2 = 2R_1$ the voltage applied across nodes A and B is halved at each step (try to show this as an exercise).

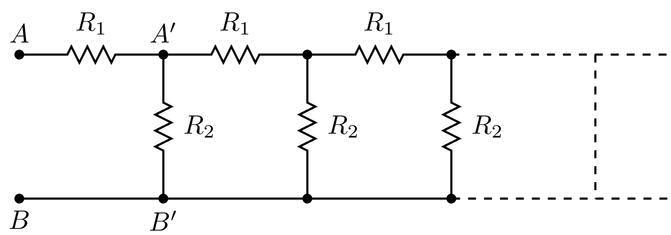
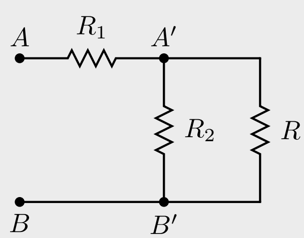


Figure 12: Infinite attenuator chain

1. Determine the input resistance R of the network. In other words, find an equivalent circuit as seen from nodes A and B such that it contains one resistor with resistance R.

The equivalent resistance between A' and B' is the resistance of the infinite chain in parallel with R_2. If the resistance of the infinite chain is R then this resistance is simply $R_2 \| R$. Then we can redraw the circuit as



We know how to find the equivalent resistance for this circuit. It is $R_1 + R_2 \| R$. Thus we get

$$R = R_1 + \frac{R_2 R}{R_2 + R}.$$

If we solve this equation for R we find

$$R = \frac{R_1 + \sqrt{R_1^2 + 4R_1R_2}}{2}.$$

[5] Exercise 4.36 in Purcell and Morin

2. A truly infinite circuit is impractical. Suggest a way to terminate the attenuator after a few steps without introducing any error in the attenuation.

We can connect a resistor with value R (derived earlier) in parallel with the last R_2 in the chain. Since R mimics the rest of the infinite chain this will not introduce any error.

Example: Consider the cube of resistors shown below where every resistor is 1 Ω. Determine the equivalent resistance between nodes A and B.

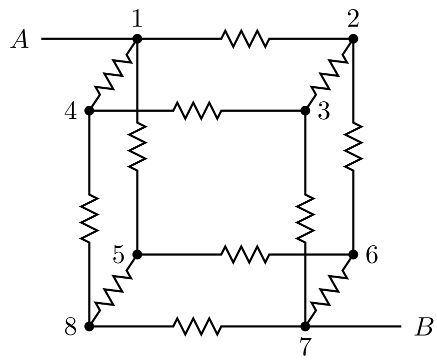
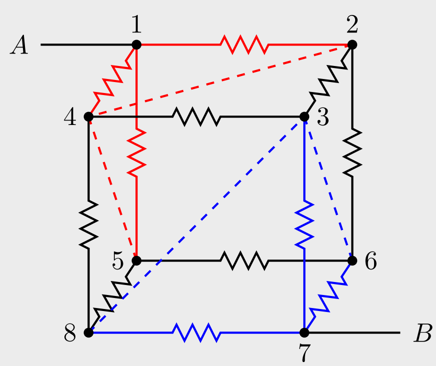
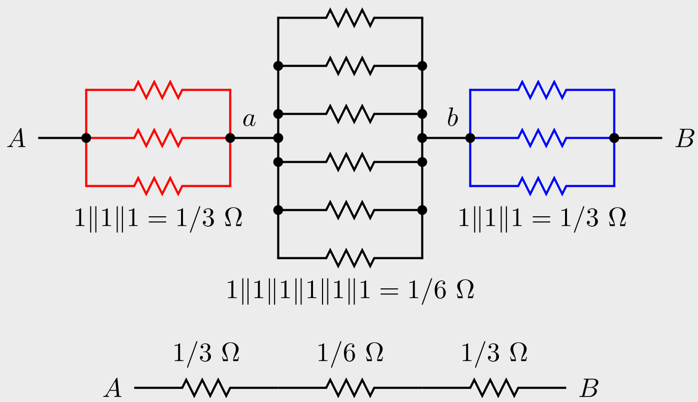


Figure 13: Cube of resistors with value 1 Ω

To solve this problem we will use a trick. We know that if two nodes are at the same voltage no current will flow, even if there is a short circuit between them. Since all the resistors are the same we know that nodes 2, 4, and 5 are all at the same potential. Similarly we know that nodes 3, 6, and 8 are all at the same potential. Since these nodes are at the same potential no current will flow if we connect them with a wire and the circuit will be equivalent.



Now if we flatten this circuit we get a circuit topology that we can simplify using the tools we have just learned. Let a be the node $\{2, 4, 5\}$ and b be the node $\{3, 6, 8\}$.



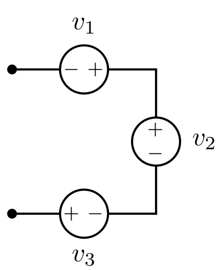
In the end we get

$$R_{eq} = \frac{5}{6}\ \Omega.$$

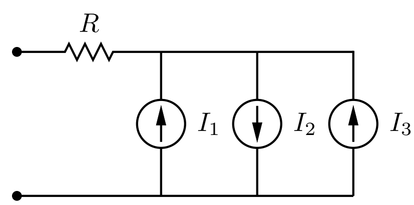
3.5.3 Sources in series and parallel

It follows from KVL that voltage sources in series add their voltages to create a single equivalent voltage source. Current sources in series are *unrealizable* meaning that it is not possible to create such a circuit. This is because a wire cannot be a two different currents simultaneously.

Similarly current sources in parallel can be summed to create a single equivalent current source. Voltage sources in parallel are unrealizable because a node must have a unique voltage.



(a) Simplifies to $v_{eq} = v_1 - v_2 + v_3$



(b) Simplifies to $i_{eq} = i_1 - i_2 + i_3$

Figure 14: Sources in series and parallel

3.5.4 Source transformation

Recall that a realistic voltage source is made from an ideal voltage source in series with a resistor and similarly a realistic current source is an ideal current source in parallel with a resistor. We now demonstrate that we can transform between realistic voltage and current sources. Consider the voltage source circuit below.

(a) Realistic voltage source (b) Realistic current source

Figure 15: For a certain choice of $R_1 = R_2$ and $i_2 = \frac{v_s}{R_1}$ these are equivalent

Application of KVL gives

$$-v_s + iR_1 + v_{12} = 0$$

which allows us to solve for i and get

$$i = \frac{v_s}{R_1} - \frac{v_{12}}{R_1}.$$

Now if we look at the current source circuit, KCL at node a gives

$$-i_s + i_{R_2} + i = 0$$

and solving for i gives

$$i = i_s - i_{R_2} = i_s - \frac{v_{12}}{R_2}.$$

Then in both circuits i and v_{12} are equivalent if

$$R_1 = R_2, \quad i_s = \frac{v_s}{R_1}.$$

Exercise:[6] Find the current I in the circuit below.

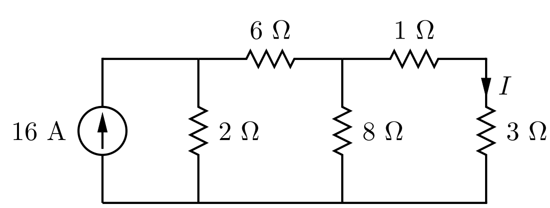


Figure 16: Circuit for exercise

Answer: $I = 2$ A.

[6] Example 2-13 in Ulaby

3.5.5 Wye-Delta transformation

Sometimes circuits are in a configuration where the resistors are neither in series nor in parallel. In this case, the Wye-Delta ($Y - \Delta$) transformation can be useful. Consider the following circuit:

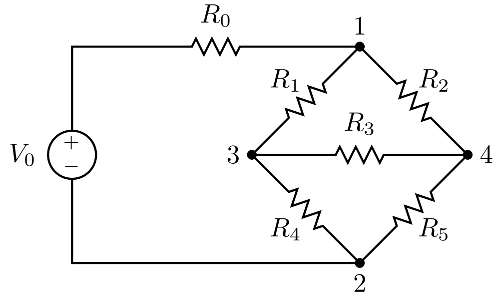
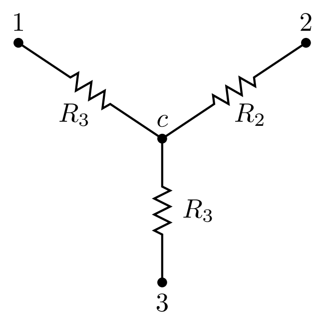
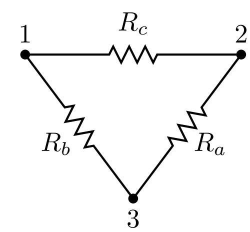


Figure 17: Series or parallel equivalence cannot be applied

No two resistors share the same current or the same voltage so we cannot apply the series or parallel equivalence developed earlier. In this case the $Y - \Delta$ transformation can help us. Consider the following two circuit configurations:



(a) Y circuit configuration



(b) Δ circuit configuration

Figure 18: Y and Δ configurations

We want to develop transformations from R_1, R_2, R_3 to R_a, R_b, R_c and vice versa such that any circuit connected at nodes 1, 2, and 3 will not be able to tell the difference between the two types of configurations. To do this we can pretend one node from the circuit is not connected to anything and then see what the transformation would be between the other nodes for both the Y and the Δ circuits.

When we let node 3 be unconnected the equivalent resistance between nodes 1 and 2 in the Y circuit is $R_1 + R_2$ because they are just connected in series. In the Δ circuit the resistor R_c is connected in parallel with the series of R_b and R_a so the equivalent resistance is $R_c \| (R_b + R_a)$. If we equate these two expressions we get an equation:

$$R_1 + R_2 = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c}.$$

The same method for the other two nodes gives two more equations:

$$R_2 + R_3 = \frac{R_a(R_c + R_b)}{R_a + R_b + R_c}$$
$$R_1 + R_3 = \frac{R_b(R_a + R_b)}{R_a + R_b + R_c}.$$

Solving these equations for R_1, R_2, and R_3 gives the desired transformation:

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$$
$$R_2 = \frac{R_a R_c}{R_a + R_b + R_c}$$
$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}.$$

Likewise we can apply the transformation in reverse to find

$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_1}$$
$$R_b = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_2}$$
$$R_c = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_3}.$$

Exercise:[7] Recall the circuit from earlier. Determine the current I.

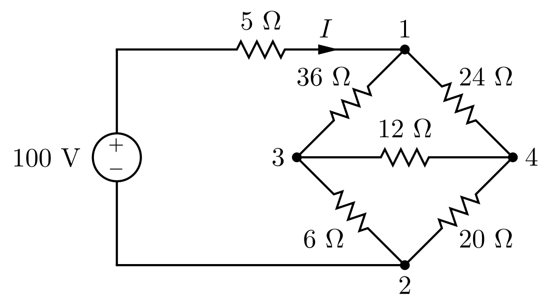


Figure 19: Circuit for $Y - \Delta$ exercise

Answer: 4 A.

3.6 Wheatstone bridge

The Wheastone bridge is a circuit that was originally developed to accurately measure the resistance of a resistor. Variations are also useful for creating sensors that can measure small changes in a resistor (such as a thermistor or a piezoresistor). The Wheatstone bridge has a similar layout to the circuit from the previous exercise, with four resistors in a diamond. The resistors R_1 and R_2 are fixed and known, R_3 is known but is variable, and R_x is the resistor we want to measure. An ammeter is connected between nodes 1 and 2.

[7]Example 2-15 in Ulaby

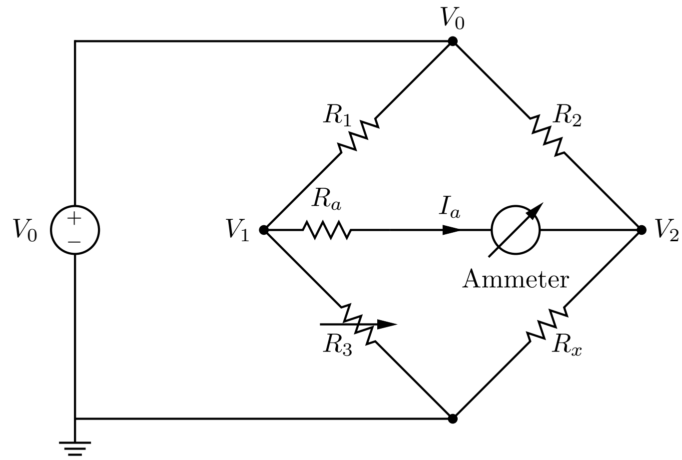


Figure 20: Wheatstone bridge circuit

To determine R_x you vary R_3 until the current I_a becomes 0. At this point the bridge is called *balanced*, and means that $V_1 = V_2$. From voltage division we know

$$V_1 = \left(\frac{R_3}{R_1 + R_3}\right) V_0, \quad V_2 = \left(\frac{R_x}{R_2 + R_x}\right) V_0.$$

Since the circuit is balanced we can set these two equal to get the equation

$$\left(\frac{R_3}{R_1 + R_3}\right) V_0 = \left(\frac{R_x}{R_2 + R_x}\right) V_0.$$

Since the voltages across R_1 and R_2 are equal we also get

$$\frac{R_1 V_0}{R_1 + R_3} = \frac{R_2 V_0}{R_2 + R_x}.$$

This equation comes from calculating the current down each branch and multiplying by the resistance to get the voltage drop across the resistor.

Dividing these two equations gives

$$\frac{R_3}{R_1} = \frac{R_x}{R_2}$$

which implies

$$R_x = \left(\frac{R_2}{R_1}\right) R_3.$$

4 Analysis Techniques

We have seen the essential for circuit analysis in the form of Ohm's law, KCL, and KVL. In this section we develop systematic methods for solving any resistive circuit through nodal and mesh analysis, as well as superposition and the important concept of Thevenin equivalence.

4.1 Node-voltage method

The node-voltage method (also called nodal analysis) is a method for using KCL to systematically solve a circuit. Problems with additional voltage sources can be solved by using a concept of a "supernode." Recall that we call a node *extraordinary* if it connects to three or more elements. The node-voltage method is the following:

1. Identify all extraordinary nodes and pick one as ground. Then assign node voltages to the remaining extraordinary nodes.
2. For each remaining extraordinary node, apply KCL for each node using the *leaving* currents for the equation.
3. Use Ohm's law to put all the equations in terms of the node voltages and the resistances (unless the current is already known from a current source).
4. Solve the system of equations to find the unknown node voltages.

For putting currents in terms of node voltages remember the passive sign convention:

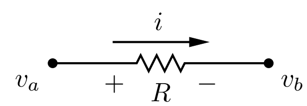


Figure 21: Current enters the "+" side, giving $i = \frac{v_a - v_b}{R}$.

For example for the following node

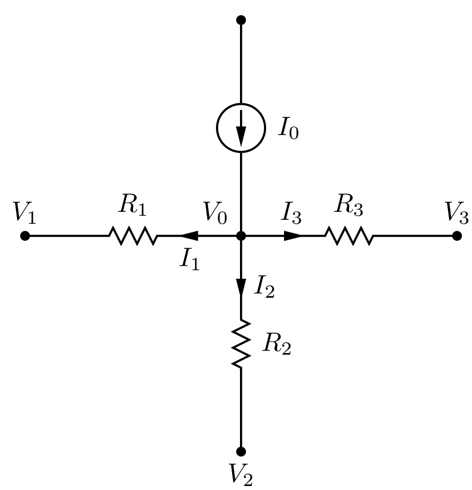


Figure 22: Node voltage

we would have

$$\frac{V_0 - V_1}{R_1} + \frac{V_0 - V_2}{R_2} + \frac{V_0 - V_3}{R_3} - I_0 = 0.$$

Example:[8] Apply nodal analysis to determine the current I in the circuit below.

[8] Exercise 3-1 in Ulaby

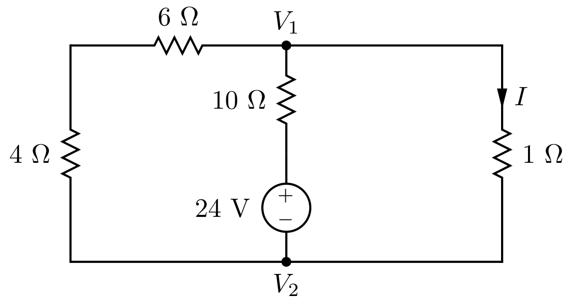


Figure 23: Determine the current I

We let node V_2 be our ground and node V_1 is the only remaining extraordinary node. We write the KCL equation for it:

$$\frac{V_1 - 0}{6+4} + \frac{V_1 - 24}{10} + \frac{V_1}{1} = 0.$$

We solve this equation and find $V_1 = 2$ V and therefore

$$I = \frac{V_1}{1} = 2 \text{ A}.$$

4.1.1 Dependent sources

Circuits containing dependent sources require no extra work. We solve them just as we would a normal circuit, and replace the value for the source with its dependency.

Example:[9] Determine I_x in the circuit shown below.

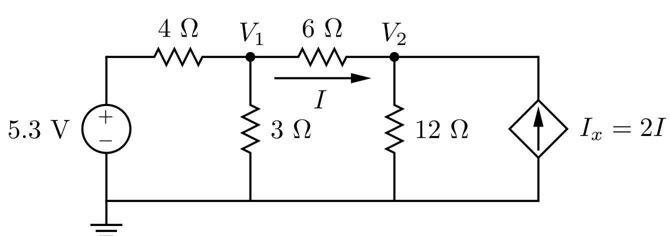


Figure 24: Determine I_x

For our two nodes V_1 and V_2 we write the KCL equations:

$$\frac{V_1 - 5.3}{4} + \frac{V_1}{3} + \frac{V_1 - V_2}{6} = 0$$
$$\frac{V_2 - V_1}{6} + \frac{V_2}{12} - I_x = 0.$$

[9]Example 3-2 in Ulaby

Now we have two equations and three unknowns. However we also know that

$$I_x = 2I = 2\left(\frac{V_1 - V_2}{6}\right).$$

When we perform this substitution we get two equations with two unknowns and we can solve the system to find

$$V_1 = 2.18 \text{ V}, \quad V_2 = 1.87 \text{ V}.$$

Hence

$$I_x = 2\left(\frac{V_1 - V_2}{6}\right) = 0.1 \text{ A}.$$

4.1.2 Supernodes

Sometimes a circuit will contain a voltage source between two extraordinary nodes with no other elements between the nodes. This creates a problem if we try to directly apply our previous method. To solve the problem we use the concept of a *supernode*.

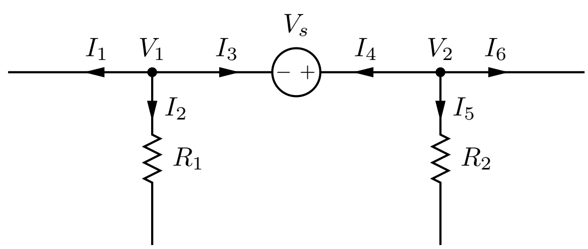
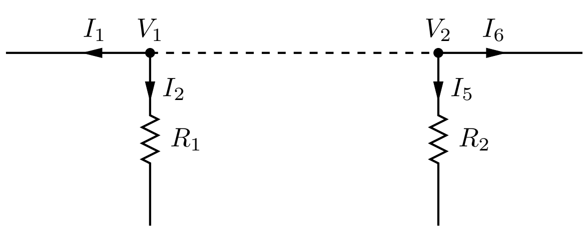
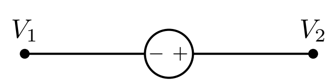


Figure 25: A supernode

A supernode is a set of two extraordinary nodes with an independent or dependent voltage source between them. The supernode may include elements such as resistors that are in parallel with the voltage source. To solve a problem with nodal analysis we can then remove the voltage source and simply connect the two nodes together because any current that enters the supernode must leave the supernode. Then we can add an auxiliary equation to relate the two node voltages, since we know the difference of the two node voltages will be the voltage given by the source.



(a) Remove source, KCL gives $I_1 + I_2 + I_5 + I_6 = 0$



(b) Auxiliary equation gives $V_2 - V_1 = V_s$

Figure 26: Simplified supernode

Example:[10]: Determine I in the circuit below.

[10]Exercise 3-3 in Ulaby

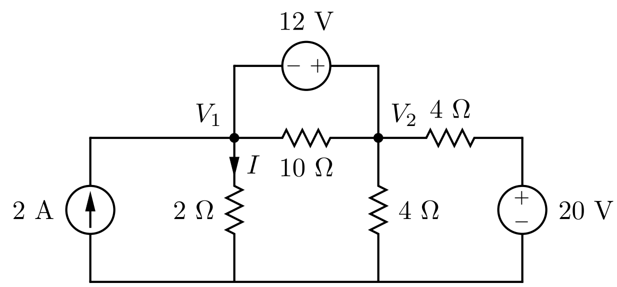


Figure 27: Circuit analysis with supernode

We make the nodes V_1 and V_2 into a supernode. KCL at the supernode gives

$$-2 + \frac{V_1}{2} + \frac{V_2}{4} + \frac{V_2 - 20}{4} = 0.$$

For the supernode we know

$$V_2 - V_1 = 12.$$

We now have two equations and two unknowns so we solve for V_1 and V_2 and get

$$V_1 = 1, \quad V_2 = 13.$$

The current I is then

$$I = \frac{V_1}{2} = 0.5 \text{ A}.$$

4.2 Mesh-current method

Mesh analysis is the dual of nodal analysis, and uses KVL and loops instead of KCL and nodes. It is good to know both methods because different problems can be more easily solved with one or the other. Nonetheless people often have a strategy that they prefer of the two.

Remember that a mesh was defined as a loop with no other loops within it. In the mesh-current method we will associate a current with each mesh. The mesh current is the current flowing through the branches of that mesh. However since circuit elements can be part of more than one mesh, it is possible that the current going through a certain element will be a combination of mesh currents.

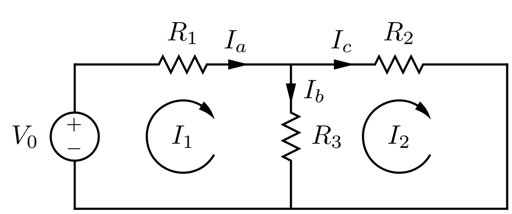


Figure 28: Mesh-current example

In the example above we have two meshes and two mesh currents I_1 and I_2. The elements R_1 and R_2 are solely part of meshes 1 and 2 respectively but R_3 is in both meshes. Thus the currents through R_1 and R_2 are

$$I_a = I_1, \quad I_c = I_2$$

and the current through R_3 is a combination of I_1 and I_2:

$$I_b = I_1 - I_2.$$

Current I_1 is positive because it is going in the direction of I_b while I_2 is going in the opposite direction so it is negative.

To actually solve the circuit above we would apply KVL at each mesh to give two equations allowing us to solve for our two unknowns I_1 and I_2:

$$\begin{aligned} I_1 &: \ -V_0 + I_1R_1 + (I_1 - I_2)R_3 = 0 \\ I_2 &: \ (I_2 - I_1)R_3 + I_2R_2 = 0. \end{aligned}$$

Note that the currents I_a, I_b, and I_c have no effect on what our KVL equations will be. Their direction tells us how to relate I_1 and I_2 sign-wise to find them once the mesh currents are known.

When writing KVL for a device that is part of two meshes, it is generally the case that the mesh current of the loop you are currently writing KVL for will be positive while all other mesh currents will be negative.

The general steps for circuit analysis with mesh analysis are

1. Identify all meshes and assign each one a mesh current. As a convention always define the mesh currents to go in the clockwise direction.
2. Apply KVL to each mesh.
3. Solve the system of equations to find the mesh currents.
4. Relate the mesh currents to the actual currents through elements in the circuit using the correct signs.

Example:[11] Determine the current I_4.

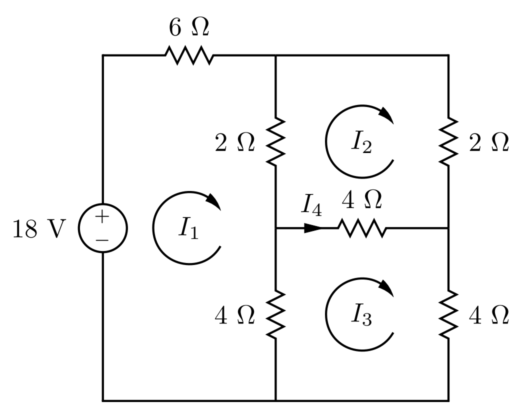


Figure 29: Mesh-current exercise

We write our three KVL equations for meshes 1, 2, and 3:

[11] Example 3-4 in Ulaby

$$
\begin{aligned}
I_1\colon &\ -18 + 6I_1 + 2(I_1 - I_2) + 4(I_1 - I_3) = 0 \\
I_2\colon &\ 2(I_2 - I_1) + 2I_2 + 4(I_2 - I_3) = 0 \\
I_3\colon &\ 4(I_3 - I_1) + 4(I_3 - I_2) + 4I_3 = 0.
\end{aligned}
$$

This gives a system of three equations and three unknowns which we can solve to get

$$I_1 = 2, \quad I_2 = 1, \quad I_3 = 1.$$

The current $I_4 = I_3 - I_2$ so

$$I_4 = 0 \text{ A}.$$

If you look closely this is actually a wheatstone bridge in the balanced condition so this result makes sense.

Just like with the node-voltage method, dependent sources do not cause problems. We just incorporate them into the equations.

4.2.1 Supermeshes

Like the supernode from nodal analysis, the supermesh is useful when there is a current source on a branch between two meshes.

4.3 Linearity and superposition

4.4 Thevenin/Norton equivalence

4.5 Maximum power transfer

5 RC and RL Circuits

We will now introduce two more devices: the capacitor and the inductor. With these two devices, the circuits we can construct become more interesting, especially from a signal processing perspective. In addition we'll now start considering voltage sources and inputs that are not DC. The source could be some arbitrary waveform that is generated by a sensor in the world or could be generated by a function generator. First we will look at basic first-order RC and RL circuits using differential equations in the time domain. We'll then extend this to second-order RLC circuits in the next section. Ultimately we will develop tools for how circuits behave in the frequency domain which will provide a fully general way to analyze circuits composed of arbitrary sets of resistors, capacitors, and inductors.

In this section we will continue using DC sources for the most part but with switches, so the sources can instead be modeled as pulse generators.

5.1 Capacitors

A conductor with charge q on it has a certain potential V with respect to a point at infinity. This electric potential is linearly related to how charged the conductor is, and the constant of proportionality is called *capacitance*:

$$q = CV.$$

The unit of capacitance is the farad (F = C/V). For example the potential of a spherical conductor with radius a is $V = q/(4\pi\epsilon_0 a)$. Thus the capacitance of the conductor is

$$C = \frac{q}{V} = 4\pi\epsilon_0 a.$$

A more interesting application of capacitance is the capacitance between two conductors charged with q and $-q$ respectively. Instead of measuring the potential with respect to infinity we measure the potential difference between the two conductors ΔV. The object composed of the two conductors is called a *capacitor*.

The most basic form of a capacitor is the *parallel-plate* capacitor. Two metal plates of area A are separated by some distance d. We know from electromagnetism that the electric field between the two plates is

$$E = \frac{\sigma}{\epsilon_0}$$

where σ is the surface charge density and we neglect the edges. Since the electric field is uniform the potential difference is simple $\Delta V = Ed$. Thus we get the equation

$$\sigma = \frac{\epsilon_0 \Delta V}{d}.$$

Since σ is the surface charge density, we know the total charge on the plate is $q = A\sigma$, where A is the area of the conductor. Thus we have

$$q = A\frac{\epsilon_0 \Delta V}{d}.$$

Now we know from the definition of capacitance that $q = C\Delta V$ so we can find that the capacitance of the parallel plate capacitor is

$$C = \frac{\epsilon_0 A}{d}.$$

Often an insulating (diaelectric) material will be placed between the two plates of the capacitor. In such a case the constant ϵ_0 (permittivity of free space) would need to change to the permittivity of the material.

Example: We have two conducting concentric cylindrical shells with height ℓ. The inner shell has radius a and outer shell has radius b. What is the capacitance of this configuration? Recall that the the electric field created by a cylinder with charge q, height ℓ, and radius r is

$$\vec{E} = \frac{q}{2\pi\epsilon_0 \ell r}\hat{\rho}$$

Also recall that the voltage difference between two conductors is the line integral of the electric field between the two

$$\Delta V = \int_C \vec{E} \cdot d\vec{r}.$$

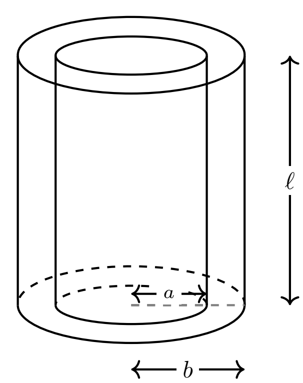


Figure 30: Concentric cylindrical shells

The voltage difference between the two cylinders is the line integral along the electric field between the two shells (since both vectors point in the same direction the dot product gives 1 and we can just work with the magnitudes).

$$\Delta V = \int_a^b E \cdot dr = \frac{q}{2\pi\epsilon_0 \ell} \int_a^b \frac{dr}{r} = \frac{q}{4\pi\epsilon_0} \ln\left(\frac{b}{a}\right).$$

Now we use the definition of capacitance: $q = C\Delta V$ to get

$$C = \frac{2\pi\epsilon_0 \ell}{\ln\left(\frac{b}{a}\right)}.$$

Many types of capacitors exist, but from now on we will simply consider capacitance to be a given.

5.1.1 Electrical properties

To develop and $i - v$ relationship for the capacitor it suffices to note that $i = dq/dt$ and then take the derivative of both sides of the definition $q = Cv$:

$$i = \frac{dq}{dt} = C\frac{dv}{dt}$$

where v is the voltage across the conductors that make up the capacitor. Just like a resistor the polarity of v and i are defined according to the passive sign convention.

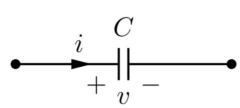


Figure 31: Current enters the "+" side, giving $i = Cdv/dt$.

By integrating both sides of the equation we can get an expression for v in terms of i:

$$\int_{t_0}^{t} \left(\frac{dv}{dt'}\right) dt' = \frac{1}{C} \int_{t_0}^{t} i(t')dt'$$

where t_0 is the initial time when the initial condition $v(t_0)$ is known. This then simplifies to

$$v(t) = v(t_0) + \int_{t_0}^{t} i(t')dt'.$$

Let's take a moment to intuitively discuss how the capacitor works. As positive charges are run through the circuit they begin to accumulate on the positive plate of the capacitor (they cannot jump the gap). As a result this positively charged plate starts to attract negative charges from the other plate, causing a current on the wire across the gap. Now remember that the voltage is a measure of how much work is necessary to move a charge between two points. As more and more charge accumulates on the positive plate, it becomes more difficult to put more positive charges there (like charges repel). Thus, the more time we have been driving current into the capacitor, the larger its voltage.

Note that this $i - v$ relationship implies the key insight that the voltage across a capacitor cannot change instantaneously, because this would imply infinite current which is impossible. This makes sense because if the voltage had change instantaneously this would mean that charge would have had to suddenly appear on the capacitor. Also if the voltage across the capacitor is constant, the device acts like an open circuit. If no charge is being put on the plates or removed (the voltage is constant) then there cannot be any current flowing.

Example:[12]: Determine the voltages V_1 and V_2 across the capacitors C_1 and C_2 in the circuit below. Assume that the circuit has been in its present (charged) condition for a long time.

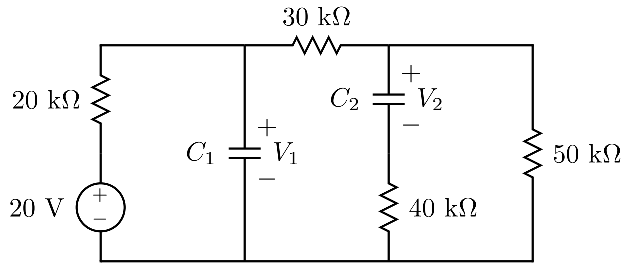
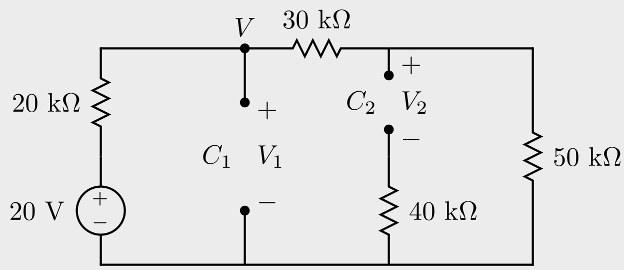


Figure 32: Capacitor circuit at DC

Since the problem says the circuit has been in its state for a long time we can assume that all voltages are DC (the capacitors have fully charged). This means that the capacitors will act like open circuits because $i = Cdv/dt = 0$, so this problem reduces to a resistive circuit problem.



We apply KCL at node V and find

$$\frac{V - 20}{20\text{k}} + \frac{V}{30\text{k} + 50\text{k}} = 0.$$

We solve and find $V = 16$ V so

$$V_1 = V = 16 \text{ V}$$

and

$$V_2 = 20 - \frac{(20\text{k} + 30\text{k})20}{20\text{k} + 30\text{k} + 50\text{k}} = 10 \text{ V}.$$

Now let's take a moment to think about the following circuit.

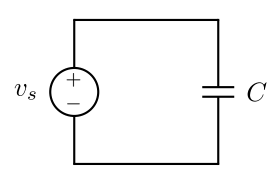


Figure 33: Capacitor connected directly to a source

[12]Example 5-4 in Ulaby

Since the voltage across a capacitor cannot change instantaneously this circuit is only realizable if v_s is a continuous waveform whose initial voltage matches the initial voltage of the capacitor. For example, suppose the capacitor begins uncharged and then is connected to a DC voltage source. Such a circuit would be unrealizable (unless the DC source is 0V). So what actually happens if you do this? Remember that a realistic voltage source is an ideal source with a small resistance. So in reality we would have the following circuit.

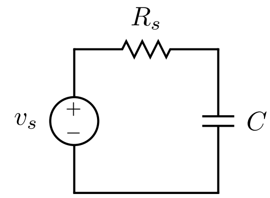


Figure 34: Capacitor connected directly to a realistic source

At the first instant the voltage across the capacitor must be zero so the resistor accounts for the entire voltage drop v_s. Then as the capacitor charges the drop across the resistor lessens until the capacitor accounts for nearly the entire drop. Meanwhile as it becomes more and more difficult to put charge on the capacitive plate the current goes to zero. This is an RC circuit and we will examine it more closely soon. In our case of connecting a DC source directly to a capacitor though we must be careful because often the resistor R_s will be very small (to model an ideal source as much as possible). This means that at the instant $t = 0$ the power dissipated across the resistor is

$$p = i_s^2 R_s = \left(\frac{v_s}{R_s}\right)^2 R_s = \frac{v_s^2}{R_s}.$$

Since v_s is likely to be large and R_s is likely to be small, it is possible that, depending on the wattage rating of the resistor, the resistor (and hence the source) will burn up.

5.1.2 Energy stored in a capacitor

A capacitor with some potential difference v creates an electric field between the two plates. This electric field is the medium for storing energy in the capacitor. Suppose we have a capacitor with capacitance C and potential difference v. We know the charge on each plate must be Cv and $-Cv$ respectively. Now if we transport a charge dq from the negative plate to the positive plate, working against the electric field the work that must be done is $dw = vdq = qdq/C$. Therefore to charge the capacitor from an uncharged state to q_f takes energy given by

$$w = \frac{1}{C}\int_0^{q_f} qdq = \frac{q_f^2}{2C}.$$

This is the energy stored in the capacitor. Since $q_f = Cv$ we can also express this as

$$w = \frac{1}{2}Cv^2$$

where v is the voltage across the capacitor at the time when the energy is being measured. This means that the energy stored in a capacitor only depends on the voltage across the capacitor at that instant.

We can also express the power being supplied by or supplying a capacitor.

$$p = iv = Cv\frac{dv}{dt}.$$

5.1.3 Series and parallel equivalence

Just like groups of resistors in series or in parallel could be simplified to one equivalent resistor, capacitors in series or parallel can also be simplified to one capacitor. Consider three capacitors in series.

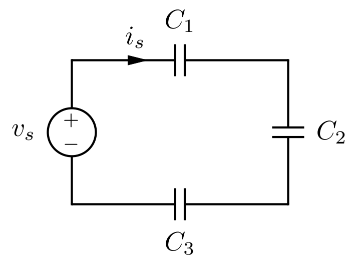


Figure 35: Capacitors in series

They share the same current i_s from the source, meaning

$$i_s = C_1 \frac{dv_1}{dt} = C_2 \frac{dv_2}{dt} = C_3 \frac{dv_3}{dt}.$$

We also know from KVL that

$$-v_s + v_1 + v_2 + v_3 = 0.$$

Now suppose we had an equivalent capacitor with capacitance C_{eq}. Then

$$\begin{aligned} i_s = C_{eq} \frac{dv_s}{dt} &= C_{eq} \left(\frac{dv_1}{dt} + \frac{dv_2}{dt} + \frac{dv_3}{dt} \right) \\ &= C_{eq} \left(\frac{i_s}{C_1} + \frac{i_s}{C_2} + \frac{i_s}{C_3} \right) \end{aligned}$$

If we solve this equation for C_{eq} we find

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right)^{-1}.$$

More generally, for any series of capacitors $C_1, C_2, \ldots C_n$ we have

$$C_{eq} = \left(\sum_{i=1}^{n} \frac{1}{C_i} \right)^{-1}.$$

Notice that this is the same formula as for resistors in *parallel*. One might be able to guess now that capacitors in parallel will act like resistors in series, but we'll still derive the result.

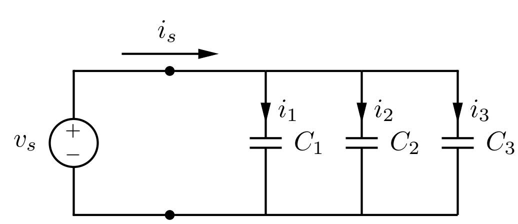


Figure 36: Capacitors in parallel

From KCL we know that

$$i_s = i_1 + i_2 + i_3 = C_1 \frac{dv_s}{dt} + C_2 \frac{dv_s}{dt} + C_3 \frac{dv_s}{dt}.$$

For an equivalent capacitor we want

$$i_s = C_{eq} \frac{dv_s}{dt}$$

so setting the two equations for i_s equal we have

$$C_{eq} \frac{dv_s}{dt} = C_1 \frac{dv_s}{dt} + C_2 \frac{dv_s}{dt} + C_3 \frac{dv_s}{dt}$$

which gives

$$C_{eq} = C_1 + C_2 + C_3.$$

More generally for any parallel combination of n capacitors $C_1, C_2 \ldots, C_n$ we get

$$C_{eq} = \sum_{i=1}^{n} C_i.$$

5.1.4 Voltage division

We saw the concept of a voltage divider that uses resistors earlier. Let's look at the same circuit topology but with capacitors.

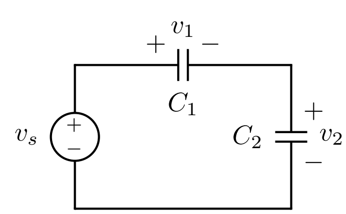


Figure 37: Capacitive voltage division

We would like to know the voltage drops v_1 and v_2 across the capacitors. We can derive the result by using the energy stored in a capacitor. The sum of the energies of the two capacitors will be the same as the energy stored in a single equivalent capacitor with a voltage of v_s across it. This means we have

$$\frac{1}{2} C_{eq} v_s^2 = \frac{1}{2} v_1 C_1^2 + \frac{1}{2} v_2 C_2^2.$$

We know

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}, \quad v_s = v_1 + v_2$$

so we can plug these values into the previous equation to get

$$\frac{1}{2} \left(\frac{C_1 C_2}{C_1 + C_2} \right) (v_1 + v_2)^2 = \frac{1}{2} v_1 C_1^2 + \frac{1}{2} v_2 C_2^2$$

which reduces to

$$C_1 v_1 = C_2 v_2.$$

If we plug in $v_2 = v_s - v_1$ then we can rearrange terms to give

$$v_1 = \left(\frac{C_2}{C_1 + C_2}\right) v_s.$$

Likewise with $v_1 = v_s - v_2$ we find

$$v_2 = \left(\frac{C_1}{C_1 + C_2}\right) v_s.$$

Example:[13]: Determine C_{eq} and $v_{eq}(0)$ across nodes a and b for the circuit below given that $C_1 = 6$ µF, $C_2 = 4$ µF, $C_3 = 8$ µF, and the initial voltages on the three capacitors are $v_1(0) = 5$ V and $v_2(0) = v_3(0) = 10$ V respectively.

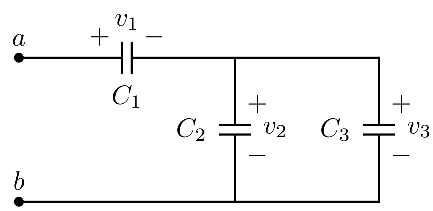


Figure 38: Capacitor example circuit

The equivalent capacitance is found by combining C_2 and C_3 in parallel and then in series with C_1:

$$C_{eq} = \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3} = 4 \text{ µF}.$$

The equivalent voltage follows simply from the fact that first we drop 5 volts across v_1 and then 10 volts down to node b.

$$v_{eq}(0) = 15 \text{ V}.$$

5.2 Response of the RC circuit

An RC circuit is a circuit composed of sources, resistors, and capacitors which can be reduced to a generic form with just a source, resistor, and capacitor in series.

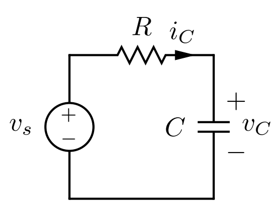


Figure 39: RC circuit

[13]Exercise 5-9 in Ulaby

In such a circuit we know that after some time if the source is at some DC voltage the capacitor will become charged with a voltage V_s across it and current will cease to flow. This is called the *steady-state* solution and describes what behavior the circuit settles into after an initial response. The initial response is called the *transient* solution and that is what we seek to understand now. With both the steady-state and transient solutions we know the full behavior of the circuit. In our analysis, the sources will be DC and we will look at the response as switches select between them.

The following circuit is the general form of the RC step response circuit.

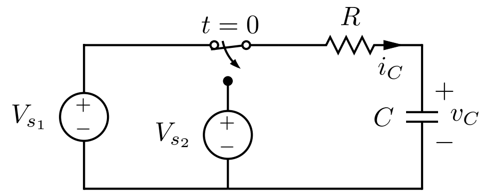
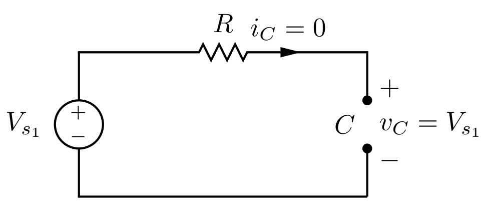


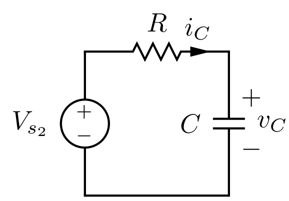
Figure 40: General RC step response circuit

The switch has been in its current state for long enough to enter the steady state. Then at some time $t = 0$ it switches to the second position. Depending on the difference between V_{s_1} and V_{s_2} the capacitor may then begin to charge or discharge (it is easiest at first to think of the case where V_{s_1} is 0 and the capacitor begins charging or where V_{s_2} is 0 and the capacitor begins discharging).

Since the voltage across the capacitor cannot change instantaneously, v_C cannot change in the instant that the switch flips, so we conclude that $v_C(0) = V_{s_1}$.



(a) Before the switch flips the circuit is at DC



(b) After the switch flips

Figure 41: Before and after the switch changes position

Once the switch has flipped the KVL equation for the loop is

$$-V_{s_2} + i_C R + v_C = 0$$

and by using $i_C = C dv_C/dt$ this equation can be rewritten as

$$\begin{aligned} -V_{s_2} + C\frac{dv_C}{dt}R + v_C = 0 \\ \frac{dv_C}{dt} + \frac{1}{RC}v_C = \frac{V_{s_2}}{RC} \\ \frac{dv_C}{dt} + av_C = b \end{aligned}$$

where $a = 1/RC$ and $b = V_{s_2}/RC$. This is a first-order differential equation and can be solved by multiplying both sides by e^{at} and integrating:

$$e^{at}\frac{dv_C}{dt} + av_C e^{at} = be^{at}.$$

Now note that

$$\frac{d}{dt}\left(v_C e^{at}\right) = e^{at}\left(\frac{dv_C}{dt} + av_C\right)$$

so this simplifies to

$$\frac{d}{dt}\left(v_C e^{at}\right) = be^{at}.$$

Integrating both sides from 0 to t yields an equation which we can solve for $v_C(t)$ and the final answer is

$$v_C(t) = v_C(0)e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right).$$

In the limit as $t \to \infty$,

$$v_C(\infty) = \frac{b}{a} = V_{s_2}.$$

The fraction b/a is equal to V_{s_2} in this case, or more generally the steady-state voltage across the capacitor in the new configuration. By defining the *time constant* $\tau = RC = 1/a$ and $v_C(\infty)$ we can rewrite the solution as

$$v_C(t) = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right)e^{-t/\tau}, \quad t \geq 0.$$

What this means is that the capacitor will either charge or discharge from some initial value $v_C(0)$ according to an exponential decay function as it asymptotically approaches a final value $v_C(\infty)$. The time constant will determine how quickly the capacitor charges or discharges. In particular

$$v_C(\tau) = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right)e^{-1}$$

and $e^{-1} \approx 0.37$ so the time constant determines the time at which v_C has discharged or charged to a third of the original voltage difference.

Example:[14]: In the circuit below $v_C(0^-) = 24$ V. Determine $v_C(t)$ for $t \geq 0$ and plot it. Also plot the energy stored in the capacitor as a function of time.

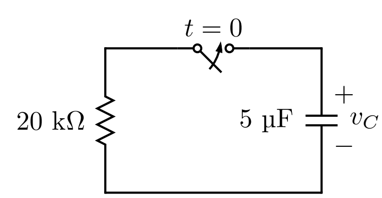
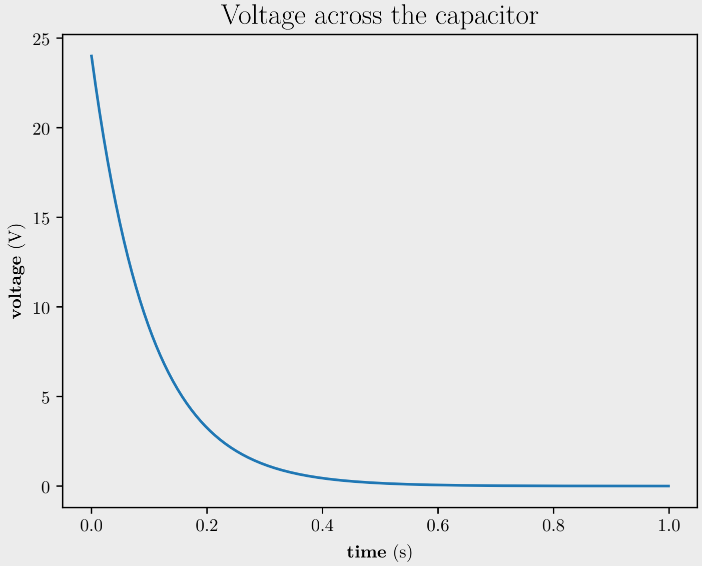


Figure 42: Discharging capacitor circuit

We know that $v_C(0) = 24$ V. The next step is to determine $v_C(\infty) = 0$ V since the capacitor will discharge completely (there is no source left to drive it). The time constant $\tau = RC = (20\text{k}\Omega)(5\mu\text{F}) = 0.1$ s. Now we simply plug into the general equation and find

[14]Exercise 5-14 from Ulaby

$$v_C(t) = 24e^{-t/0.1} = 24e^{-10t} \text{ V}.$$



The energy is

$$\frac{1}{2}Cv_C^2 = \frac{1}{2}(5\text{µF})24^2 e^{-20t} \text{ J}.$$

Example:[15] The circuit below has two switches, both of which have been open for a long time before $t = 0$. At $t = 0$ switch 1 closes and at $t = 5$ s switch 2 closes. Assume that $v_C(0) = 0$.

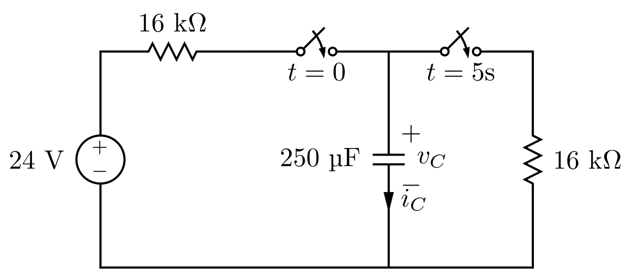


Figure 43: RC circuit for problem

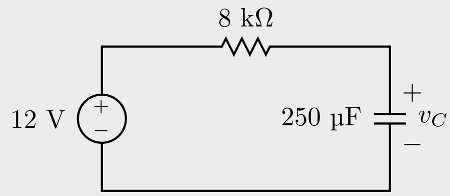
1. Determine and plot $v_C(t)$ for $t \geq 0$.

The result will be a piecewise function with one exponential for $0 \leq t \leq 5$ and another for $t \geq 5$. For the case where only the first switch is closed, $v_C(0) = 0$, and $v_C(\infty) = 24$ V. The time constant is $\tau = RC = (16\text{k}\Omega)(250\text{µF}) = 4\text{s}$. So the first equation is

$$v_C(t) = 24 - 24e^{-t/4}, \quad 0 \leq t \leq 5.$$

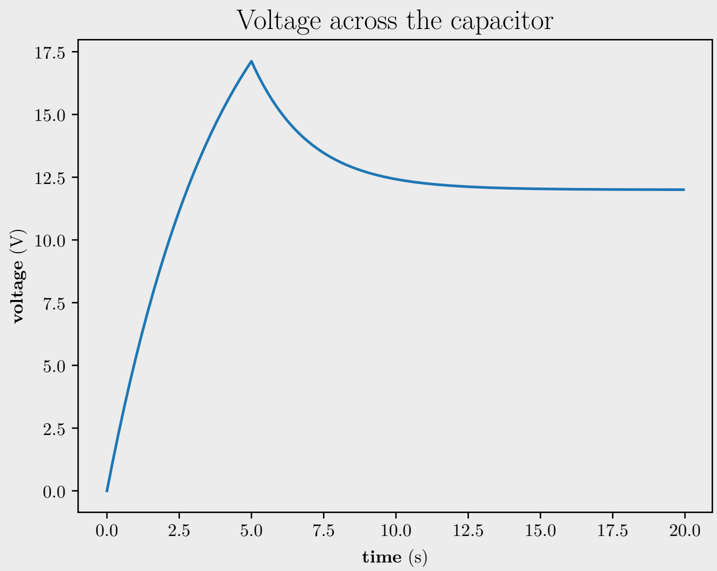
At $t = 5$s the second switch closes. The value of $v_C(\infty)$ changes, and the initial voltage is now $v_C(5)$ and the time constant should be recalculated as well. To get the circuit into the standard RC form we find the Thevenin equivalent from the perspective of the capacitor, which gives

[15] Problem 5.35 in Ulaby



The time constant is now $\tau = (8\text{k}\Omega)(250\mu\text{F}) = 2\text{s}$. The initial voltage $v_C(5) = 17.12\text{V}$ and $v_C(\infty) = 12\text{V}$. Therefore

$$v_C(t) = \begin{cases} 24 - 24e^{-t/4} & 0 \leq t \leq 5 \\ 12 + 5.12e^{-(t-5)/2} & t \geq 5. \end{cases}$$

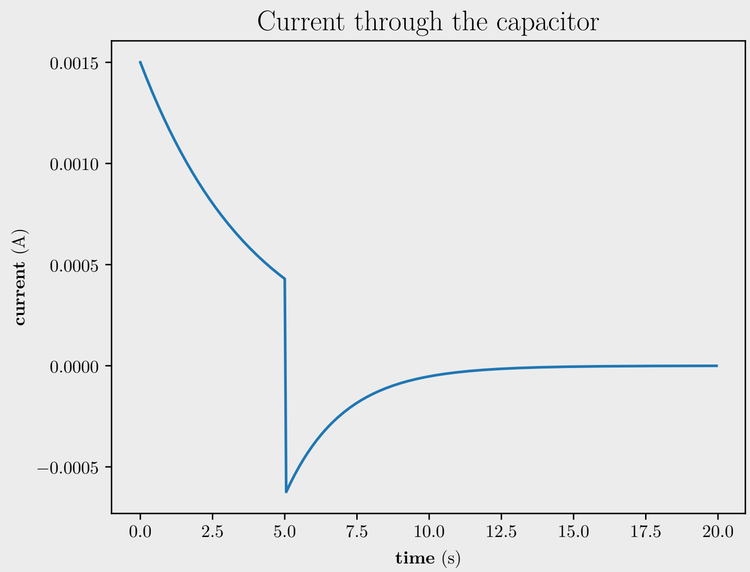


The capacitor begins charging up and then as soon as the second switch closes the resistance is increased and the voltage is decreased so the capacitor discharges down to 12V.

2. Determine and plot $i_C(t)$ for $t \geq 0$.

We already know the voltage and the current through a capacitor is

$$i_C = C\frac{dv_C}{dt} = \begin{cases} (250\ \mu\text{F})\left(6e^{-t/4}\right) & 0 \leq t \leq 5 \\ (250\ \mu\text{F})\left(-2.56e^{-(t-5)/2}\right) & t \geq 5. \end{cases}$$



This result is to be expected. Initially the switch closes and current begins flowing immediately to charge the capacitor. As the voltage across the capacitor increases the current through it decreases. Then the second switch closes and the capacitor begins discharging, quickly at first and in the opposite direction. As the capacitor reaches its steady state, the current goes to 0.

5.3 Inductors

As you may know from electromagnetism, when a current is driven through a wire, a magnetic field is created around the wire.

5.4 Response of the RL circuit

6 RLC Circuits

6.1 Series RLC

6.2 Parallel RLC

6.3 General RLC

7 Frequency Domain Analysis

7.1 Phasors

7.2 AC analysis

7.3 Impedance

7.4 Transfer functions

7.5 Frequency response

7.6 Bode plots

7.7 Filters

8 AC Power*

9 Laplace Transform Analysis*

10 Sources

This material comes from my lecture notes from ES152 taught by Prof. Todd Zickler as well as the following two books:

- *Circuit Analysis and Design* by Fawwaz Ulaby, Michel Maharbiz, and Cynthia Furse.

This is a very in-depth introduction to circuit design. It covers all the material here and more (op-amps, introductory semiconductors, Laplace transform analysis, Fourier analysis, . . .) and is very dense. I find it somewhat hard to read because of how dense it is, but it certainly has many examples. It also comes with a great companion website at http://cad.eecs.umich.edu/ complete with video lectures and self-study tests. It is freely available to all students and instructors, provided you fill out a form indicating which course you are using it for. This book was followed closely by the course and by my notes as well.

- *Electricity and Magnetism* by Edward Purcell and David Morin.

This is a classic text for learning electromagnetism. It deals a little bit with circuits, with brief sections on resistive circuits, RC, RL, and basic AC analysis. Mainly though it discusses electromagnetism and the physics behind it. If you want to go one step lower into the physics of electricity, then this is a great book.